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Dimensional Reduction of a 6-dimensional Self-Dual Gauge Field Theory

T.N. Sherry* and D.H. Tchrakian†
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

Abstract: A self-dual gauge field theory of the type introduced in refs. (5)(6) on a 6-dimensional manifold is subject to a dimensional reduction procedure to yield gauge-field-Higgs models in 4 and 3-dimensional manifolds. In the latter case, the magnetic field is defined.

Permanent address: * Dept. Math. Phys. University College Galway, Galway, Irela
† Dept. Math. Phys. St. Patrick's College, Maynooth, Ireland.

1. Introduction.

Ever since the pioneering works of 't Hooft(1) and Polyakov(2) on the monopole solutions of the Yang-Mills-Higgs system in three dimensions, and the subsequent work of Belavin et al(3) on the instanton solutions of the pure Yang-Mills system in four Euclidean dimensions, the study of finite action (energy) solutions to the field equations of gauge theories has attracted very considerable attention.

In particular it is the self-dual solutions of both the three(4) and four dimensional Euclidean gauge field theories that have yielded the most interesting results, in that they have yielded finite action (energy) solutions that are characterised by topological invariants, e.g. the Pontryagin index.

More recently, it was proposed(5)(6) that finite action Euclidean field theories with self-dual solutions could be constructed on arbitrary dimensional manifolds. On all odd dimensional manifolds, these theories would involve a gauge covariant scalar (Higgs) field in addition to the gauge field, while on all even dimensional manifolds there would only occur the purely gauge field system, and in these cases, the topological invariants would be the higher Chern classes, namely the $\frac{1}{2}N$th Chern class for a N-dimensional manifold.

Probably the only way of applying these higher dimensional theories to the construction of physical models is to use some procedure of dimensional reduction(6). This is precisely the purpose of the present paper. The actual case tackled is the gauge field theory(5) on a six dimensional manifold, which is dimensionally reduced to both four and three dimensional gauge theory models.

The plan of this paper is the following: Below, in subsections (i) and (ii) of the introduction respectively, we present

i) a summary of the dimensional reduction procedure used subsequently, and,
ii) a summary of the six-dimensional gauge field theory(5) which is our starting point.

In section 2 we present the model on a four dimensional euclidean manifold obtained from the 6-dimensional theory by reducing the dimensions from six to four; in section 3 the corresponding reduction of dimensions from six to three is performed. Section 3a is devoted to the definition of the topological flux-density.

In what follows we have restricted our attention to models resulting from the dimensional reduction of N=6 theories. The generalization of this method to even N 6 is straightforward, if a little more complicated.

(i) Dimensional Reduction.

We adopt the elementary procedure(7) used to pass from a four dimensional self-dual Yang-Mills theory to the corresponding three dimensional Yang-Mills-Higgs self-dual theory,

The procedure consists of restricting all fields $A_\mu (\mu=1,2,3,4)$, as well as the elements $g(\vec{x})$ of the gauge group, to depend only on $x_i\ (i=1,2,3)$ and be independent of the variable x_4. Then the field component A_4, which is now gauge covariant, can be identified as the Higgs field , whence it follows that the following component of the curvature

$$F_{i4} = D_i A_4 = D_i \Phi , \tag{1}$$

is just the covariant derivative of the Higgs field.

It is understood throughout this paper, that all fields concerned take their values in the algebra of SU(n), the gauge group.

It follows from the above, that the action density of the four dimensional Yang-Mills theory reduces to the three dimensional Yang-Mills-Higgs density, and that the integral inequality

$$\mathrm{tr} \int \tfrac{1}{4} F_{\mu\nu}^2 \, d_4x \geq \tfrac{1}{4}\, \mathrm{tr} \int {}^*F_{\mu\nu} F_{\mu\nu} \, d_4 x \tag{2}$$

reduces to

$$\mathrm{tr} \int \left[\tfrac{1}{4} F_{ij}^2 + \tfrac{1}{2} (D_i \Phi)^2 \right] d_3x \geq \tfrac{1}{2} \varepsilon_{ijk}\, \mathrm{tr} \int F_{jk} D_i \Phi \, d_3 x \, . \tag{3}$$

Equation (3) leads, at once, to the Bogomolnyi(4) (self-duality) equation.

(ii) The six-dimensional theory.

Here we summarise the N=6 case, of the gauge field theories proposed in ref. (5). This is not a conventional Yang-Mills theory; rather it has the following action density

$$\mathcal{A} = \mathrm{tr} \left(F_{MN}^2 + \kappa^2\, {}^{(+)}F_{MN}^2 \right) . \tag{4}$$

Here F_{MN} is the curvature $(M,N=1,\ldots,6)$, κ is a dimensional constant with the dimensions of $(\text{length})^2$, and ${}^{(+)}F_{MN}$ is the following dual field (over six dimensions)

$${}^{(+)}F_{MN} = \frac{1}{4!} \varepsilon_{MNRSTU} F_{RSTU}$$

with the four-form in (5a) given by

$$F_{MNRS} = \{F_{MN}, F_{RS}\} + \{F_{MR}, F_{SN}\} + \{F_{MS}, F_{NR}\}. \qquad (6)$$

The complementary duality operation to (5a) is

$$^{(-)}F_{MNRS} = \frac{1}{2!}\, \varepsilon_{MNRSTU}\, F_{TU}\ , \qquad (5b)$$

so that

$$F_{MN} = {}^{(+)}\left({}^{(-)}F\right)_{MN}.$$

The self-duality equations for this theory are arrived at by minimising the action absolutely i.e. requiring that the following integral inequality

$$\int \mathcal{A}\, d_6x \geqslant 2\kappa\, \mathrm{tr} \int {}^{(+)}F_{MN}\, F_{MN}\, d_6x \qquad (7)$$

achieves an equality. In that case the self-duality equations

$$F_{MN} = \kappa\, {}^{(+)}F_{MN} \qquad (8)$$

actually solve the Euler-Lagrange equations of this theory(5).

We note that the self-duality equations (8) lead trivially to an identically zero curvature for the gauge group SU(2)(5). It is important to point out, however, that this is not the case for gauge groups SU(n) when n>2.

2. A Four-dimensional model.

This model is derived from the six-dimensional self-dual gauge theory described above, by means of the dimensional reduction procedure of section 1(i).

According to this method of dimensional reduction, we invoke the indepdence of all field quantities of the variables x_5 and x_6. Then, restricting to gauge transformations whose parameters depend only on x_μ, μ=1,2,3,4, it is clear that the A_4 and A_5 components of the connection are gauge covariant fields, and so

we identify them as two Higgs fields

$$A_5 = \Phi_1 \text{ and } A_6 = \Phi_2.$$

It then follows that the following components of the curvature itself can be expressed as

$$F_{\mu 5} = D_\mu \Phi_1 \qquad (9,a)$$
$$F_{\mu 6} = D_\mu \Phi_2 \qquad (9,b)$$
$$F_{56} = [\Phi_1, \Phi_2]. \qquad (10)$$

In terms of the curvature on the four dimensional manifold, , and the two Higgs fields (=1,2), the action density (4) reduced to the following form

$$\begin{aligned} \mathcal{A}_4 = \mathrm{tr}\,[F_{\mu\nu}^2 + 2(D_\mu\Phi_1)^2 + 2(D_\mu\Phi_2)^2 + 2[\Phi_1,\Phi_2]^2] \\ + \kappa^2 \mathrm{tr}\,[\{F_{\mu\nu}, D_\rho\Phi_1\}(\{F_{\mu\nu}, D_\rho\Phi_1\} + \{F_{\nu\rho}, D_\mu\Phi_1\} + \{F_{\rho\mu}, D_\nu\Phi_1\}) \\ + \{F_{\mu\nu}, D_\rho\Phi_2\}(\{F_{\mu\nu}, D_\rho\Phi_2\} + \{F_{\nu\rho}, D_\mu\Phi_2\} + \{F_{\rho\mu}, D_\nu\Phi_2\}) \\ + \{F_{\mu\nu}, [\Phi_1,\Phi_2]\}(\{F_{\mu\nu}, [\Phi_1,\Phi_2]\} - 4\{D_\mu\Phi_1, D_\nu\Phi_2\}) \\ + 2\{D_\mu\Phi_1, D_\nu\Phi_2\}(\{D_\mu\Phi_1, D_\nu\Phi_2\} - \{D_\mu\Phi_2, D_\nu\Phi_1\}) \\ + \tfrac{1}{4}\{F_{\mu\nu}, F_{\rho\sigma}\}(\{F_{\mu\nu}, F_{\rho\sigma}\} + \{F_{\mu\rho}, F_{\sigma\nu}\} + \{F_{\mu\sigma}, F_{\nu\rho}\})]. \end{aligned} \qquad (11)$$

Complicated as this action density is, the self-dual solutions must lead to a finite action integral. The self-duality equations, which minimise the action absolutely, are obtained from (8) by applying the above dimensional reduction procedure. They are

$$F_{\mu\nu} = \kappa\, \varepsilon_{\mu\nu\rho\sigma} [\tfrac{1}{2}\{F_{\rho\sigma}, [\Phi_1,\Phi_2]\}, \{D_\rho\Phi_1, D_\sigma\Phi_2\}] \qquad (12a)$$
$$D_\mu\Phi_1 = -\tfrac{1}{2}\kappa\, \varepsilon_{\mu\nu\rho\sigma}\{F_{\rho\sigma}, D_\nu\Phi_2\} \qquad (12b)$$
$$D_\mu\Phi_2 = +\tfrac{1}{2}\kappa\, \varepsilon_{\mu\nu\rho\sigma}\{F_{\rho\sigma}, D_\nu\Phi_1\} \qquad (12c)$$
$$[\Phi_1, \Phi_2] = \tfrac{1}{4}\kappa\, \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \kappa\{{}^*F_{\mu\nu}, F_{\mu\nu}\}. \qquad (12d)$$

where ${}^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the "4-dimensional dual" of $F_{\mu\nu}$.

We now examine the consequences of the self-duality equations (12). It follows from (7) that the action takes the following form for solutions of (12):

$$\int \mathcal{A}_4\, d_4x = 2\kappa\, \mathrm{tr} \int {}^{(*)}F_{MN} F_{MN}\, d_4x \,. \tag{13}$$

Then, using (12) we find after some straightforward computations the following form for the action integral

$$\int \mathcal{A}_4\, d_4x = 3 \int \Box\, \mathrm{tr}(\Phi_1^2 + \Phi_2^2)\, d_4x \,. \tag{13'}$$

To give the final form for the action integral of the self-dual fields, we observe the following consequence of equations (12): From (12b,c) and the Bianchi identities it follows that

$$\partial_\mu \mathrm{tr}\, \Phi_1^2 = \partial_\mu \mathrm{tr}\, \Phi_2^2 \,, \tag{14}$$

which means that $\mathrm{tr}\, \Phi_1^2 - \mathrm{tr}\, \Phi_2^2$ = const., and choosing the vanishing value for the constant, we have

$$\mathrm{tr}\, \Phi_1^2 = \mathrm{tr}\, \Phi_2^2 \tag{14'}$$

Thus, in its simplest form, the self-dual action for this four-dimensional theory takes the form

$$\int \mathcal{A}_4\, d_4x = 6 \int \Box\, \mathrm{tr}\, \Phi_1^2\, d_4x \,. \tag{13''}$$

This form of the action is instructive insofar as it highlights the role of the boundary conditions to be required of self-daual solutions with finite topological action. Thus, requiring the following asymptotic form for $\mathrm{tr}\, \Phi_1^2$

$$\mathrm{tr}\, \Phi_1^2 = \text{const.} - \frac{n}{|x-x_0|^2} + O\left(|x-x_0|^2\right) \tag{15}$$

[illegible] a finite action $6(4\pi)^2 n$, with integer spectrum n modulo $6\cdot(4\pi)^2$. [illegible] solutions are found, they would play a similar role in our theory [illegible] which the instantons play in the pure Yang-Mills gauge theory. The

solutions, if they exist, are distinguished from the instanton solutions of the pure Yang-Mills gauge theory on three counts: the theory includes two Higgs fields in addition to the gauge field; the gauge group must be larger than SU(2)[5]; the use of the third Chern density rather than the Pontryagin density in the evaluation of the topological integrals.

3. A Three-dimensional Model.

This model is derived from the six-dimensional model described in section by dimensional reduction to three dimensions. The resulting Higgs fields are identified with the corresponding components of the connection field as

$$A_4 = \Phi_1 \,, \quad A_5 = \Phi_2 \,, \quad A_6 = \Phi_3 \,,$$

and hence we can identify the following components of the curvature field as covariant derivatives of Φ_α, $\alpha=1,2,3$ (note that α is not the manifold index i=1,2,3):

$$F_{i4} = D_i \Phi_1 \,, \quad F_{i5} = D_i \Phi_2 \,, \quad F_{i6} = D_i \Phi_3 \qquad (16a,b,c$$

$$F_{45} = [\Phi_1, \Phi_2] \,, \quad F_{56} = [\Phi_2, \Phi_3] \,, \quad F_{64} = [\Phi_3, \Phi_1] . \qquad (17a,b,c$$

The action density (4) then takes the form

$$\begin{aligned} \mathcal{L}_3 = {} & \mathrm{tr}\left[F_{ij}^2 + 2(D_i \Phi_\alpha)^2 + 2\Psi_\alpha^2 \right] \\ & + \kappa^2\, \mathrm{tr}\left[2\{B_i, D_i \Phi_\alpha\}^2 + \{F_{ij}, \Psi_\alpha\}^2 + 2\{D_i \Phi_\alpha, \Psi_\alpha\}^2 \right. \\ & \left. + \left(\varepsilon_{\alpha\beta\gamma}\{D_i \Phi_\beta, D_j \Phi_\gamma\}\right)^2 - 2\varepsilon_{\alpha\beta\gamma}\{F_{ij}, \Psi_\alpha\}\{D_i \Phi_\beta, D_j \Phi_\gamma\} \right] \qquad (1 \end{aligned}$$

where we have used the abbreviation $B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk}$, as well as the notation

$$\Psi_\alpha = \varepsilon_{\alpha\beta\gamma}\, \Phi_\beta \Phi_\gamma$$

and the repeated α, β, γ indices imply summation.

Next we exhibit the resulting self-duality equations arising from (8)

$$B_i = \kappa\{D_i \Phi_\alpha, \Psi_\alpha\} \qquad (19a$$

$$D_i \Phi_\alpha = \kappa\{B_i, \Psi_\alpha\} - \tfrac{1}{2}\kappa\, \varepsilon_{ijk}\, \varepsilon_{\alpha\beta\gamma}\{D_j \Phi_\beta, D_k \Phi_\gamma\} \qquad (19b$$

$$\Psi_\alpha = \kappa\{B_i, D_i \Phi_\alpha\} . \qquad (19c$$

Solutions of (19), which are regular and satisfy certain boundary conditions would lead to finite and topological values of the integral of (18) over three-dimensional euclidean space. As in the previous Section, we now exhibit the boundary behaviour of the solutions that guarantees a topological integral of (18).

Again, the action integral for self-dual solutions is

$$\int \mathcal{L}_3\, d_3x = 2K \int {}^{(+)}F_{MN}\, F_{MN}\, d_3x \tag{20}$$

and after a lengthy but straightforward computation, we show that by virtue of equations (19), the action integral (20) can be written in the form

$$\int \mathcal{L}_3\, d_3x = 2 \int \Delta\, \mathrm{tr}\,(\Phi_1^2 + \Phi_2^2 + \Phi_3^2)\, d_3x \tag{20'}$$

which in turn will be guaranteed to have integer spectrum of the field quantity $\mathrm{tr}\,\Phi_\alpha^2$ is regular and has the following asymptotic form

$$\mathrm{tr}\,\Phi_\alpha^2 = \mathrm{const.} - \frac{n}{|\vec{x}-\vec{x}_0|} + O(|\vec{x}-\vec{x}_0|^2) + \dots \tag{21}$$

In our model, solutions of this form will play the same role as that played by the solutions[4] to the Yang-Mills-Higgs system, namely, as a model for the magnetic monopole[1][2] in a non-Abelian gauge theory. Again in the present theory the gauge group will have to be larger than SU(2).

This highly non-linear gauge field model is a possible theory for the magnetic monopole. It is therefore interesting to define the topological flux-density, which is closely related to the electromagnetic field[6]. As this definition is independent of the model that we are discussing in this section, we shall devote a subsection to it.

3a. The Electromagnetic Field.

The electromagnetic field is defined by 't Hooft[1] for the Yang-Mills-Higgs system for SU(2). The flux of the magnetic field on the other hand can be obtained from the knowledge of the electromagnetic field at spatial infinity, where the flux is evaluated as a surface integral[6]. It is this flux that turns out to be proportional to the total action in the case of self-dual solutions,

as can be seen immediately from inequality (3).

From our viewpoint therefore, we seek the electromagnetic field at infinity only, for example the flux density on the right hand side of (3) after having converted the volume integral to a surface integral. This task can be performed for any gauge field theory on three dimensions. For the 't-Hooft-Polyakov monopole, it can best be presented as a dimensional reduction from the inequality (2) on a four dimensional manifold to the inequality (3) on three dimensions, where the electromagnetic flux density (at infinity) is

$$\mathcal{B}_i = \mathrm{tr}\,\Phi\, B_i\,. \tag{22}$$

Clearly $\mathcal{B}_i$ can be defined[6] as the field quantity which is obtained from the dimensional reduction of (2), the right hand side of which is expressible as the volume integral of a divergence.

This last property, that the Pontryagin and Chern classes defined as integrals of gauge invariant polynomials of the curvature are expressible as surface integrals, is the main ingredient in the definition of $\mathcal{B}_i$ for a three dimensional gauge field theory (not necessarily self-dual) which is obtained from an (even) N-dimensional pure gauge field theory.

In our present example, where N=6, the 3rd Chern density is expressed as a divergence in the following form

$$2\,\mathrm{tr}\,{}^{(+)}F_{MN} F_{MN} = \varepsilon_{MNABCD}\,\partial_M\,\mathrm{tr}\,A_N\left[F_{AB}F_{CD} - A_A A_B F_{CD} + \tfrac{2}{5} A_A A_B A_C A_D\right]. \tag{23}$$

Subjecting (23) to the dimensional reduction performed in this Section, we first find that

$$\begin{aligned} 2\,\mathrm{tr}\,F_{MN}\,{}^{(+)}F_{MN} &= \varepsilon_{iNABCD}\,\partial_i\,\mathrm{tr}\,A_N\left[F_{AB}F_{CD} - A_A A_B F_{CD} + \tfrac{2}{5} A_A A_B A_C A_D\right] \\ &= \partial_i \mathcal{B}_i \end{aligned} \tag{23'}$$

which defines $\mathcal{B}_i$, up to an added divergenceless term. This can be computed straightforwardly to give

$$\mathcal{B}_i = 8\,\mathrm{tr}\left[2 B_i \Phi_\alpha \Phi_\alpha - \tfrac{1}{2}\varepsilon_{ijl}\,\varepsilon_{\alpha\beta\gamma}\,\Phi_\alpha \{D_j \Phi_\beta, D_l \Phi_\gamma\}\right]. \tag{24}$$

The electromagnetic flux-density B for a theory derived from an (even)

N-dimensional pure gauge-field theory is completely straightforward to obtain by following the steps analogous to (23-(24). The resulting topological charges would be related to the $(\frac{N}{2})$-th Chern Classes respectively.

4. Conclusions or Discussion

In this paper we have presented two gauge (field) theories which may possess self-dual finite action (energy) solutions, one over a four-dimensional Euclidean manifold, the other over a three dimensional Euclidean manifold. These models were derived from a recently proposed[(5)(6)] non-standard gauge field theory over a six dimensional Euclidean manifold by a dimensional reduction procedure.

The original six dimensional model is unusual in that a set of self-duality equations solve the Euler-Lagrange equations of motion. The action density for the model contains both the conventional Yang-Mills term and a higher order term involving a dimensional parameter κ. The effect of the higher order term is to make the dimensionally reduced Lagrangians appear more complicated than conventional models. The dimensional parameter κ is introduced to ensure that each term in the action density possesses the correct dimensions. It also enters the self-duality equations in such a way that κ must be strictly non-zero to avoid trivial solutions.

The self-dual solutions to the gauge theories considered in this paper are to be distinguished from the conventional self-dual solutions to Yang-Mills-Higgs theories - they are not a simple generalization of the latter. That the gauge group must be larger than SU(2)[(5)] has been emphasized in the text, thus in the limit that we restrict our gauge group to an SU(2) subgroup, the solutions become trivial. On the other hand, if we take the $\kappa \to 0$ limit, so that our action densities reduce to those of conventional models, the self-duality equations trivialize. Thus we see that the solutions considered in this paper constitute a new type of classical solution - yet the three dimensional model of section 3 provides a possible model for the magnetic monopole in non-Abelian gauge theory.

In this paper we have not given any proofs of the existence of the types of solutions considered, nor have we explicitly constructed any such solutions. What we have done is to present Yang-Mills-Higgs-type models, over both four and three dimensions, which allow such interesting solutions to exist. The construction of explicit solutions of this type is being undertaken at present. It is our expectation that, when we know the form of the explicit solutions, the significance of the more unusual features of the models, namely the dimensional parameter and the associated higher order terms in the Lagrangian, will be more apparent.

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